Gyanshauryam, International Scientific Refereed Research Journal
© 2023 GISRRJ | Volume 6 | Issue 1 | ISSN : 2582-0095

# Properties of Homomorphism on Rings

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**Article Info**
**Publication Issue :**
Volume 6, Issue 1
January-February-2023
**Page Number :** 44-60
**Article History**
Accepted : 01 Jan 2023
Published : 14 Jan 2023

**ABSTRACT**

The purpose in this paper is to determine the finiteness properties of the homomorphism $C(Y) \to C(X)$ (i.e., whether it is finite, integral, singly generated or finitely generated) in terms of the properties of the map $X \to Y$.
**Keywords :** Continuous Functions, Finite Homomorphism

## Introduction

We shall see the, in order to study finiteness properties between the associated rings of continuous functions, the problem for compactifications is equivalent to the problem for arbitrary compact Hausdorff spaces. The main result of the paper says that, for X and Y compact Hausdorff spaces, the homomorphism $C(Y) \to C(X)$ is finite (C(X) is finitely generated as C(Y)-module) if and only if the map $X \to Y$ is locally injective. We show examples of finite but not singly generated, as well as integral but not finite, homomorphisms $C(Y) \to C(X)$. Every continuous map $X \to Y$ defines, by composition, a homomorphism $C(Y) \to C(X)$ between the corresponding algebras of real-valued continuous functions. This paper is devoted to the study of the finiteness properties of this homomorphism. Our starting point is the well-known result which states that every real compact space $X$ is determined by the algebra $C(X)$ of all real-valued continuous functions defined on it, and that continuous maps between their algebras of continuous functions. From this equivalence it follows that statements about topological properties of spaces and maps should have a natural translation in terms of algebraic properties of the corresponding algebras and homomorphisms.

**Finite spaces-** In this section we study the simplest case of the problem: when the space Y is finite. First, we investigate the finiteness properties of C(X) as an $\mathbb{R}$-algebra, i.e., of the homomorphism $\mathbb{R} = C(\{p\}) \to C(X)$ given by a constant map $X \to Y = \{p\}$.

**Proposition 1.** The following conditions are equivalent for a completely regular (Hausdroff) space $X$:

(1) $X$ is a finite space.

(2) $C(X)$ is a finite $\mathbb{R}$-algebra.

(3) $C(X)$ is an integral $\mathbb{R}$-algebra.
(4) $C(X)$ is a singly generated $\mathbb{R}$-algebra.
(5) $C(X)$ is a finitely generated $\mathbb{R}$-algebra.

**Proof.** (1) ⇒(2) If $X = \{x_1, x_2, \ldots, x_n\}$, then C(X) = C({x1}) $\oplus \ldots \oplus$ C($\{x_n\}$) = $\mathbb{R} \oplus \ldots \oplus \mathbb{R} = \mathbb{R}^n$.
(2) ⇒(3) Every finite ring homomorphism is integral.
(3) ⇒(1) If every function $f \in$C(X) is a root of a polynomial with real coefficients, then $f$ (X) is finite, and then so os X, because in an infinite completely regular space we can always define a real continuous function with infinite range.
(1) ⇒ (4) We shall see that C(X) = $\mathbb{R}[f]$, for any function $f$ that separates points in $X = \{x_1, x_2, \ldots, x_n\}$. Set $\lambda_i = f(x_i)$, and take

$$f_i = \frac{\prod_{i \neq j}(f - \lambda_j)}{\prod_{i \neq j}(\lambda_i - \lambda_j)}, \quad \text{for } 1 \leq i \leq n \text{ .}$$

Certainly$f_i \in \mathbb{R}[f]$. Moreover, $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for $\neq j$ . Finally observe that, for any $\boldsymbol{g} \in$ C$(X)$, $\boldsymbol{g} = \sum_{i=1}^{n} \boldsymbol{g}(\boldsymbol{x}_i) \boldsymbol{f}_i \in \mathbb{R}[f]$.
(4) ⇒(5) this follows from the definitions.
(5) ⇒(1) The number of minimal prime ideals of a finitely generated $\mathbb{R}$-algebra is finite (because it is a noetherian ring). In $C(X)$ each prime ideal is contained in a unique maximal ideal. Hence, if $C(X)$ is a finitely generated $\mathbb{R}$-algebra, then it has only finitely many maximal ideals, and so the space X is finite.

**Corollary 2.** Let $\pi: X \to Y$ be a continuous map between compact Hausdorff spaces. If the induced homomorphism C$(Y) \to$ C$(X)$ is finite (integral, singly generated or finitely generated), then each fibre $\pi^{-1}(y)$ is a finite set.

**Proof.** By Tietze's extension theorem, $C(\pi^{-1}(y))$ is a quotient ring of $C(X)$. This implies that if C$(Y) \to$ C$(X)$ is finite (integral, singly generated, finitely generated), then so is $\mathbb{R} =$ C$(y) \to C(\pi^{-1}(y))$.

**Corollary 3.** Let Y be a finite space, and $\pi: X \to Y$ be a continuous map. The following cinditions are equivalent:

(1) $X$ is a finite space.
(2) C$(Y) \to$ C$(X)$ is a finite.
(3) C$(Y) \to$ C$(X)$ is an integral.
(4) C$(Y) \to$ C$(X)$ is a singly generated.
(5) C$(Y) \to$ C$(X)$ is a finitely generated.

**Proof.** (1) ⇒(2), (1) ⇒(3), (1) ⇒(4) and (1) ⇒(5) If $C(X)$ is a finite )integral, singly generated or finitely generated) $\mathbb{R}$-algebra, then it has the same property as $C(Y)$-algebra

The converse follows directly from 2.

**Finite homomorphisms**

**Proposition 1.** Let $\pi: X \rightarrow Y$ be a continuous map between real-compact spaces. If the homomorphisms $C(Y) \rightarrow C(X)$ is finite, hten the continuous extension of $\pi$ to the Stone- Cech compactifications, $\beta\pi : \beta X \rightarrow \beta Y$, is a locally injective map.

**Theorem 2.** Let $\pi: X \rightarrow Y$ be a continuous map between compact Hausdroff spaces. The homomorphisms $C(Y) \rightarrow C(X)$ is finite if and only if the map $\pi: X \rightarrow Y$ is locally injective.

**Proof.** Suppose that $\pi: X \rightarrow Y$ is locally injective. Every point $x \in X$ has a co-zero neighbourhood U such that $\pi$ is injective on $\overline{U}$, the closure of U. Then $C(\overline{U}) \simeq C\left(\pi(\overline{U})\right)$ and, by Tirtze's extension theorem, the homomorphisms $C(Y) \rightarrow C\left(\pi(\overline{U})\right) \simeq C(\overline{U})$ is subjective.

The space X can be covered b a finite number of these co-zero sets, $X = coz(g_1) \cup \cdots \cup coz(g_n)$. Since $coz(g_i) = coz(g_i^2)$, we can take $g_i \geq 0, \forall_i$. The functions $h_i = g_i/(g_1 + \cdots + g_n)$, $i = 1, \ldots, n$, generated $C(X)$ as a $C(Y)$-module: for every $f \in C(X)$, there exist functions $f_1, \cdots, f_n \epsilon C(Y)$ such that $f = f_i$ in $coz(g_i)$, i.e., $g_i \cdot f = f_i \cdot g_i$, so $(\sum g_i)f = f_1 \cdot g_1 + \cdots + f_n \cdot g_n$ and $f = f_1 \cdot h_1 + \cdots f_n \cdot h_n$.

**Proposition 3.** Let $\pi: X \rightarrow Y$ be a continuous map between real-compact spaces. If the homomorphisms $C(Y) \rightarrow C(X)$ is finite, then $\pi$ is a closed map and the space X can be covered by a finite number of co-zero sets, $X = coz(g_1) \cup \cdots \cup coz(g_n)$, such that $\pi$ is injective on each closure $\overline{coz(g_i)}$. Consequently, $|\pi^{-1}(y)| \leq n \; \forall \; y \in Y$.

**Proof.** Assume that the homomorphisms $\phi : C(Y) \rightarrow C(X)$ $(\phi(f) = f \circ \pi)$ is finite. According to 1, the map $\beta\pi : \beta X \rightarrow \beta Y$ is locally injective, so that $\beta X$ may be covered by a finite number of co-zero dets such that $\beta\pi$ is injective on each closure, and obiously the same happens for X and $\pi$.

Next we shall prove that $\pi$ is a closed map. Frist of all, take into account that $\beta\pi : \beta X \rightarrow \beta Y$ is closed map. If we show that $\beta\pi$ transforms $\beta X - X$ into $\beta Y - Y$, then $X = \beta\pi^{-1}(Y)$, and so $\pi = \beta\pi|_X : X \rightarrow Y$is also a closed map.

In order to prove that $\beta\pi$ carries $\beta X - X$ into $\beta Y - Y$, we are going to describe the space $\beta X$ and the map $\beta\pi$ in erms of prime ideals in $C(X)$.

Let Spec $C(X)$ be the prime spectrum of the ring $C(X)$, that is, the set of prime ideals in $C(X)$ endowed with the Zariski (or hull-kernel) topology. Recall that the subspace Max $C(X)$ of Spec $C(X)$ consisting of all maximal ideals in $C(X)$ is just $\beta X$.

Each prime ideal in $C(X)$ is contained in an unique maximal ideal and the map $r_X$ : Spec $C(X) \rightarrow$ Max $C(X)$ that sends each prime ideal in $C(X)$ to the unique maximal ideal in $C(X)$ containing it, is a continuous retraction.

The map between the prime spectra Spec $C(X) \rightarrow$ Spec $C(Y)$ that sends each prime ideal P in $C(X)$ to the prime ideal $\phi^{-1}(P) = \{f \in C(Y) : \; f \circ \pi \in P\}$ is also a continuous map. The restriction of this map to $\beta X$, is just $\beta\pi: \beta X \rightarrow \beta Y$.

Given a point$p \in \beta X$, let $M_p$ be the corresponding maximal ideal in $C(X)$ According to the above description of $\beta\pi$, the maximal ideal in $C(Y)$ corresponding to the point $q = \beta\pi(p)$ is just $r_Y(\phi^{-1}(M_p))$, that is, $M_q = r_Y(\phi^{-1}(M_p))$. As the homomorphism $\phi : C(Y) \to C(X)$ is finite, $\phi^{-1}(M_p)$ is a maximal ideal in $C(Y)$, and so $M_q = r_Y\left(\phi^{-1}(M_p)\right) = \phi^{-1}(M_{p)}$. The homomorphism induced by $\phi$ between the quotient fields $C(Y)/M_q \to C(X)/M_p$ is injective and also finite. Therefore, $C(X)/M_p$ is an algebraic extension of $C(Y)/M_q$. Moreover, the field $C(X)/M_p$ is totally ordered and $C(Y)/M_q$ is real-closed, that is to say, it has no proper algebraic extensions to an ordered field. Hence, the homomorphism $C(Y)/M_Q \to C(X)/M_p$ is an isomorphism. This implies that $C(Y)/M_q = \mathbb{R}$ or, equivalently, $q \in Y$ if and only if $C(X)/M_p = \mathbb{R}$, i.e., $p \in X$.

The converse of this result is true for normal spaces Y.

**Theorem 4.** Let $\pi: X \to Y$ be a continuous map between realcompact spaces and suppose that Y is a normal space. The homomorphism $C(Y) \to C(X)$ is finite if and only if $\pi$ is a closed map and the space X can be covered by a finite number of co-zero sets, $X = coz(g_1) \cup \cdots \cup coz(g_n)$, such that $\pi$ is injective on each closure $\overline{coz(g_i)}$.

**Proof.** The proof is entirely analogous to 2.

The following example shows that the converse of 1 is not true, and that Theorem 2 does not hold for non-compact spaces.

**Example.** A continuous injective map $\pi: X \to Y$ such that $\beta\pi : \beta X \to \beta Y$ is a homomorphism and the homomorphism $C(Y) \to C(X)$ is not finite.

Let $\sum = \mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N} - \mathbb{N}$. $\Sigma$ is a realcompact and normal space, and $\mathbb{N}$ is dense and $C^*$-embedded in $\Sigma$. Therefore, $\beta\mathbb{N} = \beta\Sigma$. The homomorphism $C(\Sigma) \to C(\mathbb{N})$ induced by the inclusion map $\mathbb{N} \to \Sigma$ is nit finite, because $\mathbb{N}$ is not closed in $\Sigma$.

A class of locally injective continuous maps of especial interest consists of unbranched coverings, classically studied in connection with the fundamental group. Unbranched finite covering (locally injective, open and closed continuous maps with finite fibres) and branched finite coverings (open and closed continuous maps with finite fibres) are characterized in terms of rings of continuous functions: A continuous map between topological manifolds $\pi : X \to Y$ is an unbranched finite covering (respectively, a branched finite covering) if and only if the induced homomorphism $C(Y) \to C(X)$ is finite and flat (respectively, integral and flat).

We have as yet obtained no analogues of Theorem 2 or 4 for integral, singly generated or finitely generated homomorphisms. However, we do have some partial results. In a paper in preparation, we prove that if the homomorphism $C(Y) \to C(X)$ induced be a continuous map $\pi: X \to Y$, between compact Hausdroff spaces, is singly generated, then it is finite, and consequently the map $\pi$ is locally injective. The converse of this result is not true.

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